ROTATION NUMBERS OF DISCONTINUOUS ORIENTATION-PRESERVING CIRCLE MAPS

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Abstract. We extend a few well-known results about orientation preserving homeomorphisms of the circle to orientation preserving circle maps, allowing even an infinite number of discontinuities. We define a set-valued map associated to the lift by filling the gaps in the graph, that shares many properties with continuous functions. Using elementary set-valued analysis, we prove existence and uniqueness of the rotation number, periodic limit orbit in the case when the latter is rational, and Cantor structure of the unique limit set when the rotation number is irrational. Moreover, the rotation number is found to be continuous with respect to the set-valued extension if we endow the space of such maps with the Haussdorff topology on the graph. For increasing continuous families of such maps, the set of parameter values where the rotation number is irrational is a Cantor set (up to a countable number of points).

1. Introduction

While the well-established theory of orientation preserving homeomorphisms of the circle goes back to the works of Poincaré and Denjoy, not much has been rigourously proved for orientation preserving circle maps with discontinuities, a natural generalization of homeomorphisms

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of the circle. In real applications, continuity is often too restrictive. For example, in the study of a neuron model called Integrate-and-Fire, we are concerned with the dynamics of a map φ, that gives the time of the first spike following a given spike. When the model is periodically forced, it turns out that $\varphi(t+1) = \varphi(t) + 1$ and φ is increasing (see [5]), but not necessarily continuous. It is therefore helpful to generalize the results from homeomorphisms of the circle to discontinuous orientation-preserving circle maps. This issue was addressed in [4] for the case when there is a single discontinuity, though some of the results were not rigourously proved.

Before we go on, we will need a few definitions. A *circle map* is a map $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. A *lift* of ϕ is a real function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) = f(x) + 1$ for all x and $\pi \circ f = \phi \circ \pi$, where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical projection. A circle map ϕ is said to *preserve orientation* if it has an increasing lift (strictly). To give a better idea of what an orientation preserving circle map is, we may equivalently define this property as follows: given an orientation of the circle, ϕ preserves orientation if it maps any set of points to their image without changing their order, as shown in figure 1.

We will first prove the following theorem:

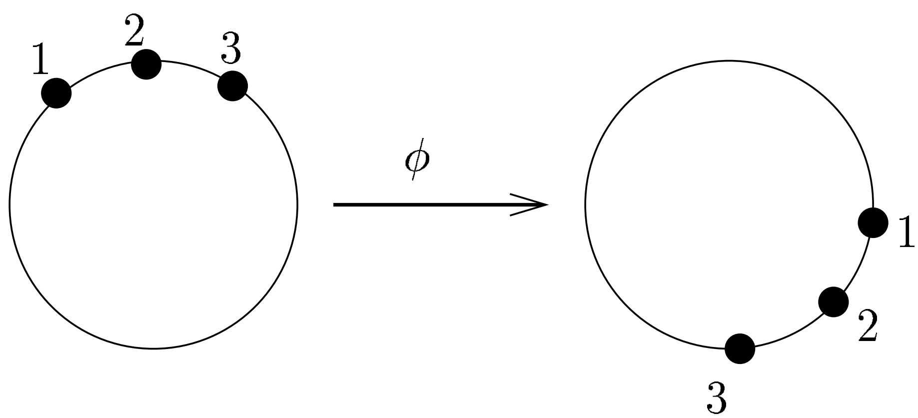
Theorem 1. *Let ϕ be an orientation preserving circle map and f an increasing lift of ϕ.*

- *The limit*

$$\lim_{n \to +\infty} \frac{f^n(x)}{n} = \alpha$$

exists and does not depend on $x \in \mathbb{R}$. This limit is called the rotation number.

FIGURE 1. An orientation preserving circle map



- *If $\alpha \in \mathbb{Q}$, then all orbits under ϕ tend to a periodic orbit with the same period.*
- *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then all orbits have the same limit set, which is either the circle or a Cantor set.*

Note that this theorem holds true even if ϕ has an infinite number of discontinuities. A proof of this theorem for homeomorphisms of the circle may be found in [2]. The first point is already known, as the proof for homeomorphisms of the circle only uses the fact that the lift f is increasing. For the second point, the usual proof relies on continuity, and for the third point, it relies on the fact that ϕ maps arcs to arcs, that is, on the intermediate value theorem. Therefore, the proofs for orientation preserving homeomorphisms of the circle do not extend naturally to orientation preserving circle maps. The idea in this paper is to extend ϕ to a set-valued map that is in some sense continuous, so that usual proofs naturally extend.

Consider the graph of a discontinuous lift f and fill the gap at each discontinuity. What we get is the graph of a *set-valued map*, that is, a map F from $\mathbb{R}$ to the subsets of $\mathbb{R}$, which we write $F : \mathbb{R} \rightsquigarrow \mathbb{R}$. Formally, this set-valued extension is defined as follows:

$$F(x) = [f(x^-), f(x^+)]$$

that is, $F(x)$ equals $\{f(x)\}$ at each point where f is continuous, otherwise it is the gap at x. Now consider the dynamical system $x_{n+1} \in F(x_n)$. The behaviour of this system is the same as the original one, except when we reach a discontinuity, where we may choose any value in the gap. Thus every orbit under f is also an orbit under F. We will find it easier to study this new dynamical system. Indeed, this set-valued map is continuous in some sense, that is, it has a closed and connected graph and shares with continuous functions a version of the intermediate value theorem. The extension F projects to a set-valued circle map Φ that maps arcs to arcs and has compact graph. Therefore proofs for homeomorphisms naturally extend to this setting.

With the same approach, we will study families of orientation preserving circle maps. If we endow the space of set-valued extensions of increasing lifts with the Haussdorff topology on their graph, which is weaker than uniform topology, then we find that the rotation number is continuous with respect to the extension. This was already proved in [7], but our method leads to a significantly shorter and less technical proof. Then a continuous family of orientation preserving circle maps is precisely a family f_t such that $(t, x) \mapsto F_t(x)$ has a closed graph and is bounded on bounded sets. For increasing families, we prove the following theorem:

Theorem 2. *Let $t \mapsto f_t$, $t \in [a, b]$, be a family of increasing discontinuous lifts such that $(t, x) \mapsto f_t(x)$ is increasing with respect to each variable and $t \mapsto F_t$ is continuous with the Haussdorff topology on the* *graph of F_t, where F_t is the set-valued extension of f_t. Let $\alpha(t)$ be the rotation number of f_t. Then*

(1) *α is continuous and non-decreasing*
(2) *for all $p/q \in \mathbb{Q} \cap \mathrm{Im}\alpha$, $\alpha^{-1}(p/q)$ is an interval containing more than one point, unless it is $\{a\}$ or $\{b\}$.*
(3) *α reaches every irrational number at most once*
(4) *α takes irrational values on a Cantor subset of $[a, b]$, up to a countable number of points.*

A restriction of the third point was proved in [7].

We shall first recall some basic ideas and results from set-valued analysis in section 2. Then we will prove theorem 1 in section 3 and theorem 2 in section 4.

2. Fundamentals of set-valued analysis

In the subsequent study we will only need very elementary results from set-valued analysis. We advise the reader to refer to [1, 8, 3] for a detailed account of set-valued analysis. All the topological spaces considered here are assumed to be metric spaces. Some results in this section have not been published before, though they are not difficult, so we shall provide the reader with a brief proof.

In the following, we shall say a set-valued map $F : X \rightsquigarrow Y$ is compact/closed when the graph of F is compact/closed in the product space $X \times Y$.

2.1. **Continuity of set-valued maps.**

We shall define here a notion of continuity for set-valued maps through the following proposition:

Proposition 1. *Let $F : X \rightsquigarrow Y$ be a set-valued map. The following assertions are equivalent:*

(1) *F is closed and the image of any conditionnally compact set by F is conditionnally compact.*
(2) *F is closed and the image of any compact set by F is compact.*
(3) *For any compact set $K \subset X$, $F_{|K}$ is compact*
(4) *For any sequence $x_n \in X$ converging to x, any sequence $y_n \in F(x_n)$ has a cluster point $y \in F(x)$.*

In this proposition, $F_{|K}$ is the restriction of F to the compact K, i.e., the set-valued map $F_{|K} : K \rightsquigarrow Y$ that equals F on K. If F is single-valued, these assertions are equivalent to continuity. We shall note $\mathcal{K}(X,Y)$, or simply $\mathcal{K}$ if there is no ambiguity, the set of set-valued maps satistying these properties. The proof of this proposition is easy and is left to the reader. It follows from property (4) of the proposition above that if $F \in \mathcal{K}(X,Y)$ and $G \in \mathcal{K}(Y,Z)$, then $G \circ F \in \mathcal{K}(X,Z)$.

The following result generalizes a well-known result for continuous single-valued maps:

Proposition 2. *Let $F \in \mathcal{K}(X,Y)$ taking connected values (i.e., $F(x)$ is connected for all $x \in X$). If $A \subset Dom\, F$ is connected, then $F(A)$ is connected.*

Let A be a connected subset of $Dom\, F$, and suppose $B = F(A)$ is not connected. Let $U_1 \subset Y$ and $U_2 \subset Y$ be a disconnection of B. Define

$$V_1 = \{x \in X, F(x) \subset U_1\} = X \setminus F^{-1}(Y \setminus U_1)$$

and V_2 in the same way. It can be seen that F^{-1} maps closed sets to closed sets, so that V_1 and V_2 are open. We have $V_1 \cap V_2 = \emptyset$ by construction. We shall prove that

$$V_1 \cap A = F^{-1}(U_1) \cap A$$

and the same equality for V_2. Obviously enough $V_1 \cap A \subset F^{-1}(U_1) \cap A$. Suppose the equality does not hold. Then there is $x \in A$ such that $F(x) \cap U_1 \neq \emptyset$ but $F(x)$ is not included in U_1. But since $F(x) \subset U_1 \cup U_2$, we must have $F(x) \cap U_2 \neq \emptyset$, which is impossible because $F(x)$ is connected. It follows that $A \subset V_1 \cup V_2$, $V_1 \cap A \neq \emptyset$ and $V_2 \cap A \neq \emptyset$, so that V_1 and V_2 are a disconnection of A, which is a contradiction.

We shall need the two following useful results:

- Let

$$\begin{array}{rcl} I \times X & \rightsquigarrow & Y \\ (t, x) & \mapsto & F_t(x) \end{array}$$

  a set-valued map in $\mathcal{K}(I \times X, Y)$, and let $K \subset X$ be a compact set. Then the set-valued map $t \mapsto F_t(K)$ is in $\mathcal{K}(I, Y)$.
- Let F_λ a family of set-valued maps in $\mathcal{K}(X, Y)$. Then the set-valued map

$$F : x \mapsto \bigcap_\lambda F_\lambda(x)$$

  is in $\mathcal{K}(X, Y)$.

The first result follows from the fact that $t \mapsto F_t(K)$ is the composition of $(t, x) \mapsto F_t(x)$ and $t \mapsto \{t\} \times K$, which are both in $\mathcal{K}$. The second result follows from property (3) in proposition (1) and from the observation that the graph of F is the intersection of the graphs of F_λ.

2.2. **Set-valued extension of a monotone function.** If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then we define its set-valued extension F as

$$F(x) = [f(x^-), f(x^+)]$$

Note that F^{-1} is a non-decreasing continuous single-valued function with plateaus. Then one can easily see that $F \in \mathcal{K}$, and $F(x)$ is a compact interval for all x.

The last result we need is a set-valued version of the intermediate value theorem and follows immediately from proposition 2:

Theorem 3. *Let $F \in \mathcal{K}(\mathbb{R}, \mathbb{R})$, such that $F(x)$ is a compact interval for all x. Then for every (x, a) and (y, b) from the graph of F and every real c between a and b, there is $z \in F^{-1}(c)$ lying between x and y.*

This property is obviously unchanged by composition.

Last of all, we will define a few convenient notations. If A and B are two subsets of $\mathbb{R}$, then the sum $A + B$ is defined naturally as

$$A + B = \{a + b | a \in A, b \in B\}$$

In the same way, $\frac{A}{n}$ is the set of real numbers $\frac{a}{n}$ with $a \in A$, and we shall write $A + x$ to mean $A + \{x\}$.

We shall write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. Thus if F is the set-valued extension of an increasing function f, then $x < y$ implies $F(x) < F(y)$. Note that if F is the extension of an increasing lift of an orientation preserving circle map, then $F(x + 1) = F(x) + 1$ for all x and F projects to a compact set-valued circle map Φ that is orientation preserving.

3. The rotation number

Let ϕ be an orientation preserving circle map and f an increasing lift of ϕ. Denote F the set-valued extension of f and Φ its projection on the circle. From now on, we consider the extended dynamical system

$$x_{n+1} \in F(x_n)$$

and its projection on the circle

$$x_{n+1} \in \Phi(x_n)$$

A *forward orbit* under Φ (resp. F) is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_{n+1} \in \Phi(x_n)$ (resp. $x_{n+1} \in F(x_n)$) for all n. A *backward orbit* under Φ (resp. F) is a forward orbit under Φ^{-1} (resp. F^{-1}). We will often use the term *orbit* to mean forward orbit when there is no ambiguity.

3.1. **Existence and uniqueness of the rotation number.** The following proposition implies the first point of theorem 1:

Proposition 3. *If $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ is the set-valued extension of an increasing lift of an orientation preserving circle map, then for any sequence $x_{n+1} \in F(x_n)$, the following limit*

$$\lim_{n\to+\infty} \frac{x_n}{n} = \alpha$$

exists and does not depend on the choice of the sequence. This limit is called the rotation number *of F.*

The proof in e.g. [2] applies almost directly to this case, but we shall provide the reader with a different and easier proof because we will use later the ideas involved in it. The proof is based on an idea of H. H. Rugh [9].

Let

$$K_n = \frac{F^n - id}{n}(\mathbb{R})$$

where id is the identity $x \mapsto x$. Because $F^n - id$ is periodic, we have $K_n = \frac{F^n - id}{n}([0,1])$, and it follows from proposition 2 that K_n is a compact interval for all n. We shall prove that any sequence $\alpha_n \in K_n$ converges to a unique limit α, which is enough to prove proposition 3.

Let $0 \leq x < y < 1$. Since F^n is increasing, we have $F^n(x) < F^n(y) < F^n(x) + 1$ for any integer n, and therefore $0 < F^n(y) - F^n(x) < 1$. It follows that the length of K_n is not greater than $\frac{1}{n}$, which implies the uniqueness of α. The inclusion

$$F^{nm} - id \subset \sum_{k=0}^{m-1} (F^n - id) \circ F^{nk}$$

holds for any integers n and m. Note that, unlike in the single-valued case, the equality needs not hold because for a non-empty set $A \subset \mathbb{R}$, we only have $0 \in A - A$, but not $A - A = \{0\}$. Dividing both sides by nm, it follows that $K_{nm} \subset K_n$ for all n and m because K_n is convex. Therefore $K_{nm} \subset K_n \cap K_m$. Thus $K_n \cup K_m$ is a compact interval with length smaller than $\frac{1}{n} + \frac{1}{m}$. It follows that any sequence $\alpha_n \in K_n$ is Cauchy, therefore converges. This common limit α is called the *rotation number.*

The rotation number can also be characterized in the following way:

Lemma 1. *We have*

$$\{\alpha\} = \bigcap_{n \in \mathbb{N}} K_n$$

where α is the rotation number.

Indeed, because the length of K_n tends to 0, we have $d(\alpha, K_n) \to 0$ as $n \to +\infty$. Now choose $n \in \mathbb{N}$. We have $d(\alpha, K_{nm}) \to 0$ as $m \to +\infty$. It follows from $K_{nm} \subset K_n$ that $d(\alpha, K_n) = 0$. Since K_n is compact, $\alpha \in K_n$.

As noted in [6], proposition 3 is not true in general if f is discontinuous and only non-decreasing.

3.2. **Rational rotation number.** To prove the second point of theorem 1, we will need the following proposition, which is well-known for homeomorphisms of the circle:

Proposition 4. *$\alpha \in \mathbb{Q}$ if and only if Φ has a periodic point, i.e., there exist an integer n and a point of the circle x such that $x \in \Phi^n(x)$.*

This proposition was proved in [6] in a somewhat different formulation. Lemma 1 makes this proposition trivial. Indeed let $\alpha = p/q$. By lemma 1, we have $p/q \in K_q$, i.e., there is $x \in \mathbb{R}$ such that $p \in F^q(x) - x$, so that $x \in \Phi^q(x)$.

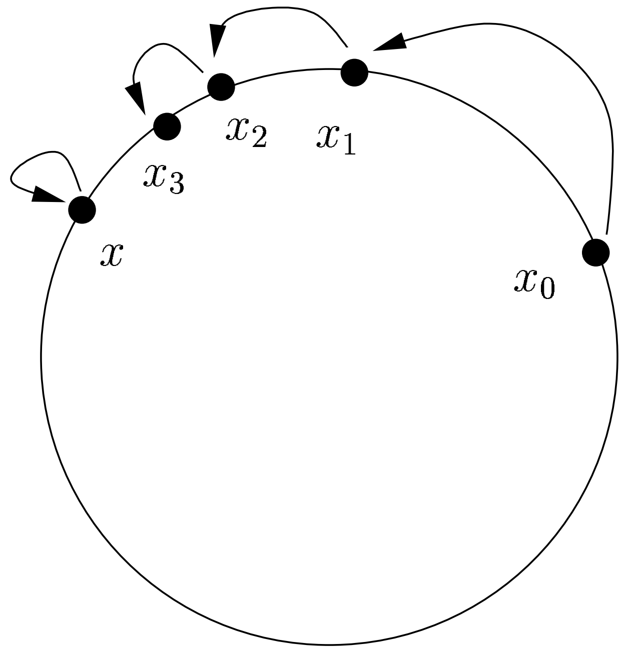
The second point of theorem 1 follows from the proposition below:

Proposition 5. *If $\alpha \in \mathbb{Q}$, then all orbits under Φ tend to a periodic orbit with the same period* [1].

If $\alpha = p/q$, then Φ^q has a fixed point x. Now consider a sequence $x_{n+1} \in \Phi^q(x_n)$. The proof is illustrated by figure 2. Consider the points x, x_0 and x_1, and choose an orientation on the circle so that x_1 lies in the arc from x to x_0. The points x, x_1 and x_0 are sent respectively to x, x_2 and x_1. Since Φ preserves orientation, it follows that x_2 lies in the arc from x to x_1. By induction, we can see that (x_n) is a decreasing sequence in the arc from x to x_0, and therefore converges to a limit x^* (not necessarily x). Therefore, for any orbit $x_{n+1} \in \Phi(x_n)$ and for every $k \in \{0, \ldots, k-1\}$, the sequence x_{nq+k} converges to a point x_k^*. Because Φ is closed, we have $x_{k+1}^* \in \Phi(x_k^*)$ (where $k+1$ is modulo q). Thus the limit points x_k^* form a q-periodic orbit under Φ.

[1] i.e., there is an integer q such that if (x_n) is an orbit under Φ, there is a q-periodic orbit (y_n) such that $d(x_n, y_n)$ tends to 0 as n tends to $+\infty$.

FIGURE 2. Proof of proposition 5



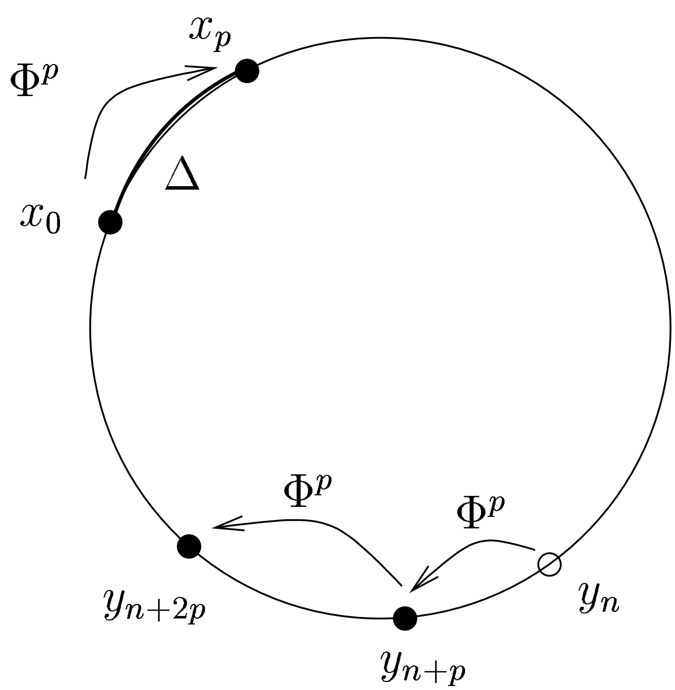
3.3. **Irrational rotation number.** Here we consider the case when the rotation number α is irrational. The key to prove the third point of theorem 1 is the following lemma, similar to a lemma for homeomorphisms in [2]:

Lemma 2. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $x_p \in \Phi^p(x_0)$ and Δ an arc with ending points x_0 and x_p. Then any forward or backward orbit under Φ intersects Δ.*

To prove this lemma, we only need the property that Φ preserves orientation. Let (y_n) be a forward orbit under Φ, and consider the sequence of points $y_n, y_{n+p}, y_{n+2p} \ldots$ Define an orientation on the circle so that Δ is the arc from x_0 to x_p. The point y_{n+p} must lie in the arc from y_n to x_p, otherwise Φ^p would map the arc from x_0 to y_n to an arc included in it, which would imply that Φ^p has a fixed point.

Suppose $y_{n+p} \notin \Delta$. Then, because Φ^p is orientation preserving, y_{n+2p} must lie in the arc from y_{n+p} to x_p (see figure 3). Going this way, we

FIGURE 3. Proof of lemma 2



can see that, as long as the sequence does not intersect Δ, we have y_{n+mp} lying in the arc from $y_{n+(m-1)p}$ to x_0. If Δ was never reached, then the sequence y_{n+mp} would converge, thus Φ^p would have a fixed point, which contradicts the fact that α is irrational. Thus there is an integer m such that $y_{n+mp} \in \Delta$. The same reasoning applies for the backward orbit, which proves the lemma.

Now we can prove the third point of theorem 1. The limit set of a sequence is its set of cluster points.

Proposition 6. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then all orbits under Φ have the same limit set Ω, which is either the circle or a Cantor set.*

Consider two orbits $x_{n+1} \in \Phi(x_n)$ and $y_{n+1} \in \Phi(y_n)$, and let x^* be a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$, i.e., x^* is the limit of a subsequence (x_{n_k}). For each k, consider the smaller arc Δ_k with endpoints x_{n_k} and $x_{n_{k+1}}$. The length of Δ_k tends to 0 as k goes to infinity. From lemma 2, we can find an increasing integer sequence m_k such that $y_{m_k} \in \Delta_k$. It follows that the sequence y_{m_k} tends to x^*. In the same way, we can find a subsequence from a backward orbit $y_n \in \Phi(y_{n+1})$ converging to x^*. Thus forward and backward orbits under Φ all have the same limit set Ω.

To prove that Ω is either the circle or a Cantor set, we shall use the following proposition, the proof of which is left to the reader:

Proposition 7. *Let X be a compact metric space and $\Phi : X \rightsquigarrow X$ a compact set-valued map. Let $(x_n)_{n\in\mathbb{N}}$ be an orbit under Φ and Ω its set of cluster points.*

Then for any $x \in \Omega$, there is a forward and backward orbit under Φ starting from x and staying in Ω. In other words, for any $x \in \Omega$, we have:

$$\Phi(x) \cap \Omega \neq \emptyset$$

$$\Phi^{-1}(x) \cap \Omega \neq \emptyset$$

Note that in the present case, since Φ^{-1} is single-valued, we have $\Phi^{-1}(\Omega) = \Omega$.

The limit set Ω is compact by definition. Let $x \in \Omega$. It follows from proposition 7 that its backward orbit is included in Ω, and since x is a cluster point of this orbit, it is not isolated in Ω. Now suppose Ω contains an open set U, and let $x_0 \in U$. We can find $n \in \mathbb{N}$ and a point $x_n \in \Phi^n(x_0) \cap U$. We shall prove that Ω is the whole circle. Indeed, for any point $y \in \mathbb{R}/\mathbb{Z}$ we can find a point $y_m \in \Phi^m(y)$ that lies in the arc with ending points x_0 and x_n included in U, by lemma 2. We have $y_m \in \Omega$, and since $y = \Phi^{-m}(y_m)$, it follows that $y \in \Omega$. Thus Ω is the whole circle. Otherwise, Ω has empty interior, and since it is a compact set with no isolated point, it is a Cantor set.

4. Families of orientation preserving circle maps

In this section we will prove theorem 2, which we recall here:

Theorem 2 *Let $t \mapsto f_t$, $t \in [a,b]$, be a family of increasing discontinuous lifts such that $(t,x) \mapsto f_t(x)$ is increasing with respect to each variable and $t \mapsto F_t$ is continuous with the Haussdorff topology on the graph of F_t, where F_t is the set-valued extension of f_t. Let $\alpha(t)$ be the rotation number of f_t. Then*

(1) *α is continuous and non-decreasing*
(2) *for all $p/q \in \mathbb{Q} \cap \mathrm{Im}\alpha$, $\alpha^{-1}(p/q)$ is an interval containing more than one point, unless it is $\{a\}$ or $\{b\}$.*
(3) *α reaches every irrational number at most once*
(4) *α takes irrational values on a Cantor subset of $[a,b]$, up to a countable number of points.*

By *Cantor set*, we mean a compact set with empty interior and no isolated point.

4.1. **Continuity of the rotation number.** Consider a family f_t of increasing lifts of orientation preserving maps, and note F_t their set-valued extension. We require a condition of regularity. Uniform continuity is too restrictive, as it would not allow the points of discontinuities to change. Instead, as the graph of F_t, restricted to a single period, is compact, we may endow the space of such set-valued maps with the

Haussdorff topology and require $t \mapsto F_t$ to be continuous in this topology. This makes for instance the family $t \mapsto F(\cdot + t)$ continuous. As F_t^{-1} is a single-valued continuous map, this condition is equivalent to requiring $t \mapsto F_t^{-1}$ to be uniformly continuous. We will actually use an equivalent condition which is much more convenient: the set-valued map $(t, x) \mapsto F_t(x)$ is in $\mathcal{K}$.

The rotation number is then found to be continuous with respect to the parameter:

Proposition 8. *Let f_t be a family of increasing lifts of orientation preserving maps, indexed on a metric space I, and F_t their set-valued extension. Assume that the set-valued map*

$$\begin{array}{rcl} I \times \mathbb{R} & \rightsquigarrow & \mathbb{R} \\ (t, x) & \mapsto & F_t(x) \end{array}$$

is in $\mathcal{K}$. Then the map $t \mapsto \alpha(f_t)$ is continuous.

This proposition is well-known in the case when f_t is continuous. In the present case, results from section 2 will provide us with an easy proof. Define $K_n(t) = \frac{F_t^n - id}{n}([0,1])$, and recall from lemma 1 that $\alpha(f_t) = \cap K_n(t)$. The set-valued map $(t, x) \mapsto F_t^n(x)$ is in $\mathcal{K}$ for all n (by composition). Then the map $(t, x) \mapsto \frac{F_t^n(x) - x}{n}$ shares the same property for all n. It follows that $t \mapsto K_n(t)$ is also in $\mathcal{K}$ for all n, so that, by intersection, the single-valued map $t \mapsto \alpha(f_t)$ is in $\mathcal{K}$, which means it is continuous.

Although proposition 8 restricts to families of lifts, it implies that the rotation number is continuous with respect to the extension of the lift in the Haussdorff topology on the graph. Indeed, this follows from proposition 8 if we take I to be the compact space $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$: then

$(n, x) \mapsto F_n(x)$ is in $\mathcal{K}$ means $Graph(F_n)$ converges to $Graph(F_{+\infty})$ in the Hausdorff topology, and $n \mapsto \alpha(f_n)$ is continuous means $\alpha(f_n)$ tends to $\alpha(f_{+\infty})$.

4.2. **Increasing families of lifts.** Here we shall prove theorem 2 announced in the introduction. Recall that we consider a family of increasing discontinuous lifts f_t of orientation preserving circle maps, indexed on an interval $[a, b]$, such that $(t, x) \mapsto f_t(x)$ is increasing with respect to each variable and $t \mapsto F_t$ is continuous with the Haussdorff topology on the graph of F_t. This means that $(t, x) \mapsto F_t(x)$ is in $\mathcal{K}$.

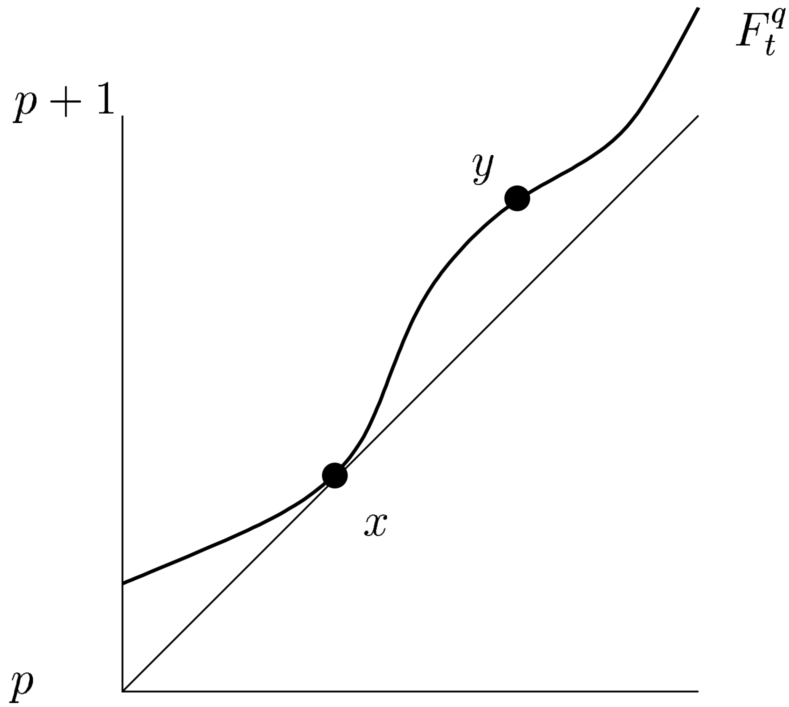
Note that the theorem would not hold without the hypothesis f_t *discontinuous*, because then the circle maps may be conjugated with rotations even for rational rotation numbers, for example: $f_t(x) = x + t$ (points 2 and 4 of the theorem do not hold).

Denote $\alpha(t)$ the rotation number of f_t. We shall prove each point of theorem 2:

(1) We already proved that α is continuous. By induction, for $s > t$, we have for all x and t: $f_s^n(x) > f_t^n(x)$. It follows that $\alpha(s) \geq \alpha(t)$, i.e., α is non-decreasing.

(2) Let $p/q \in \mathbb{Q} \cap \mathrm{Im}\,\alpha$. Because α is continuous and non-decreasing, $\alpha^{-1}(p/q)$ is a closed interval. Let $t \in \alpha^{-1}(p/q)$. We know from the proof of proposition 4 that there is an x such that $x + p \in F_t^q(x)$. Since f_t^q is not continuous, we cannot have $F_t^q = id + p$, so there is another point $y \notin F_t^q(y) - p$ (see figure 4). Suppose $F_t^q(y) - y - p$ contains a positive real (the other case is treated in the same way). Now choose $h > 0$. We have $F_{t-h}^q(x) - x - p < 0$ by monotonicity (i.e., all reals in this set are negative). If $F_{t-h}^q(y) - y - p$ contains non-negative numbers,

FIGURE 4. $\alpha^{-1}(p/q)$ contains more than one point



then by theorem 3, there is a z such that $0 \in F_{t-h}^q(z) - z - p$, so that $\alpha(t-h) = p/q$. Otherwise, again by theorem 3, there is a $u \in]t-h, t[$ such that $0 \in F_u^q(y) - y - p$, so that $\alpha(u) = p/q$. It follows that $\alpha^{-1}(p/q)$ is a closed interval containing more than one point.

(3) Let $t \in [a, b]$ such that $\alpha(t)$ is irrational. We shall prove that we can find a $u \in [a, b]$ arbitrarily close to t such that $\alpha(u)$ is rational, since it implies that α cannot reach $\alpha(t)$ twice. The extension F_t projects to a set-valued circle map Φ_t. Let y be in the limit set of Φ_t and note $x^* = \Phi_t^{-1}(y)$. Since y is also a cluster point of the backward orbit of x^*, for any $h > 0$, either the arc $[y, y+h]$ or $[y-h, y]$ (once the circle is oriented) intersects the backward orbit of x^*. Suppose the former case (the latter case

is treated similarly) and choose $h > 0$. Define the set-valued map

$$\begin{aligned} \Psi : \mathbb{R}/\mathbb{Z} &\rightsquigarrow \mathbb{R}/\mathbb{Z} \\ \Psi(x) &= \bigcup_{u \in [t,t+h]} \Phi_u(x) \end{aligned}$$

which is in $\mathcal{K}$ and $\Psi(x)$ is a closed arc for all x by proposition 2. It follows from the hypothesis of monotonicity and theorem 3 that for any $n \in \mathbb{N}$:

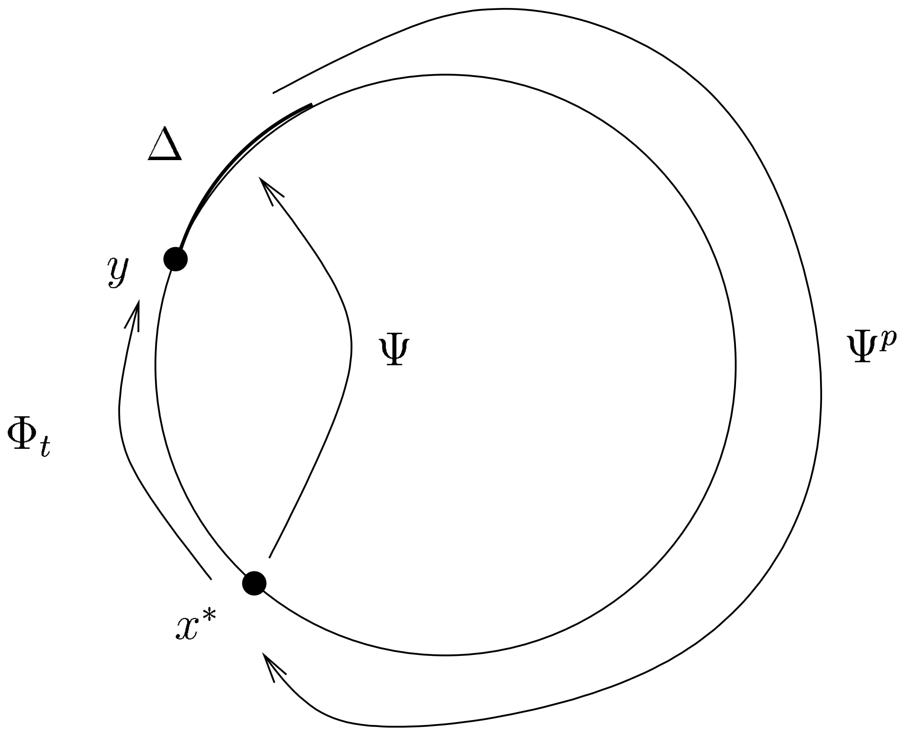
$$\Psi^n(x) = \bigcup_{u \in [t,t+h]} \Phi_u^n(x)$$

Denote $\Delta = \Psi(x^*)$, which is an arc containing y (see figure 5). It follows that for some integer p, $\Phi_t^{-p}(x^*) \in \Delta$, so that $x^* \in \Psi^p(\Delta)$, i.e., $x^* \in \Psi^{p+1}(x^*)$. This means precisely that for some $u \in [t, t+h]$, $x^* \in \Phi_u^{p+1}(x^*)$, so $\alpha(u)$ is rational. Therefore α reaches any irrational number at most once.

(4) The set $K = [a, b] \setminus \mathrm{Int}\, \alpha^{-1}(\mathbb{Q})$ differs from $\alpha^{-1}(\mathbb{R} \setminus \mathbb{Q})$ by countably many points. K is compact. From the continuity of α, none of its points is isolated. Suppose it contains an open interval J. Since $\alpha(J)$ is not reduced to a single point, it must contain a rational number p/q. But this is impossible because $\alpha^{-1}(p/q)$ is a closed interval containing more than one point. Thus K has empty interior, and since it is compact and has no isolated point, it is a Cantor set.

Note that the theorem we just proved does not need any regularity requirements apart from continuity of the family.

FIGURE 5. α reaches every irrational number at most once



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REFERENCES

[1] Aubin, J. P. and H. Frankowska. *Set-valued analysis.* Birkhäuser, 1990.

[2] Cornfeld, I. P., S. V. Fomin and Ya. G. Sinai. *Ergodic theory (Grundlehren der mathematischen Wissenschaften number 245).* Springer-Verlag, 1981.

[3] Deimling, K. *Multivalued Differential Equations.* De Gruyter, Berlin, 1992.

[4] Keener, J. P. Chaotic behavior in piecewise continuous difference equations. *Trans. Amer. Math. Soc.*, 261:589–604, 1980.

[5] Keener, J. P., F. C. Hoppensteadt and J. Rinzel. Integrate-and-fire models of nerve membrane response to oscillatory input. *SIAM J. Appl. Math.*, 41:503–517, 1981.

[6] Rhodes, F. and C. L. Thompson. Rotation numbers for monotone functions on the circle. *J. London Math. Soc.*, 34:360–368, 1986.

[7] Rhodes, F. and C. L. Thompson. Topologies and rotation numbers for families of monotone functions on the circle. *J. London Math. Soc.*, 43:156–170, 1991.

[8] Rockafellar, R. T. and R. Wets. *Variational Analysis.* Springer Verlag, 1997.

[9] Rugh, H. H. Personal communication.

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